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LETTER TO THE EDITOR

Soliton solutions of two bidirectional sixth-order partial differential equations belonging to the KP hierarchy

C Verhoeven and M Musette

Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail: mmusette@vub.ac.be and cverhoev@vub.ac.be

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Abstract

In this letter, we analyse two bidirectional sixth-order partial differential equations, which are reductions in (1 + 1) dimensions of equations belonging to the KP hierarchy. They have fourth-order and fifth-order Lax pairs, respectively. We derive their Bäcklund transformations and, from the nonlinear superposition formula, we can build their soliton solutions like a Grammian. The interesting dynamics of these solitons is that they may describe not only the overtaking collision but also the head-on collision of solitary waves of different type and shape.

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1. Reductions of two-component bilinear systems belonging to the KP hierarchy

The two integrable partial differential equations (PDEs) studied in this letter are reductions in (1 + 1) dimensions of equations belonging to the KP hierarchy [1].

We consider the following two-component bilinear system [2]

$$\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)(\tau_0 \cdot \tau_0) = 0 \tag{1}$$

$$\left(\left(D_1^3 + 2D_3\right)D_2 - 3D_1D_4\right)(\tau_0 \cdot \tau_0) = 0 \tag{2}$$

as well as

$$\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)(\tau_0 \cdot \tau_0) = 0 \tag{3}$$

$$\left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2\right)(\tau_0 \cdot \tau_0) = 0. \tag{4}$$

Here, the subscripts of the bilinear operators correspond to the components of the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ while τ_0 is a function of $\vec{x}$ satisfying the condition

$$\tau_0(x_1, x_2, x_3, x_4, \ldots) = \tau_0(x_1, -x_2, x_3, -x_4, \ldots) \tag{5}$$

imposed by the C_∞ Lie algebra symmetry, and possessing the Taylor expansion with respect to the even variables $x_2, x_4, \ldots$

$$\tau_0(\vec{x}) = f(x_{\mathrm{odd}}) + \tfrac{1}{2}x_2^2 g(x_{\mathrm{odd}}) + \cdots. \tag{6}$$

From the system (1)–(2), assuming that τ_0 is independent of x_4, we obtain the following bilinear system in $(1 + 1)$ dimensions

$$\left(D_1^4 - 4D_1D_3\right)(f \cdot f) + 6fg = 0 \tag{7}$$

$$\left(D_1^3 + 2D_3\right)(f \cdot g) = 0 \qquad f = \tau_0|_{x_2=x_4=\cdots=0} \qquad g = \left.\frac{\partial^2\tau_0}{\partial x_2^2}\right|_{x_2=x_4=\cdots=0} \tag{8}$$

which is equivalent to the coupled KdV-type PDEs ($x_3 \equiv t, x_1 \equiv x$) [1–4]

$$-4u_t + 12uu_x + u_{xxx} + 3v_x = 0 \qquad u = \frac{\partial^2}{\partial x^2}\operatorname{Log} f \tag{9}$$

$$2v_t + 6uv_x + v_{xxx} = 0 \qquad v = g/f \tag{10}$$

and, by elimination of v, to the sixth-order scalar equation ($u = z_x$) [5]

$$-8z_{tt} + z_{xxxxxx} - 2z_{xxxt} + 18z_x z_{xxxx} + 36z_{xx}z_{xxx} + 72z_x^2 z_{xx} = 0. \tag{11}$$

This integrable PDE, which is of second order in time, is here called the bidirectional Satsuma–Hirota (bSH) equation.

The similarity reduction $z = V(x) - \frac{\kappa}{16}t^2$ of equation (11) gives

$$V^{(5)} + 18V'V''' + 9(V'')^2 + 24(V')^3 + \kappa x + \beta = 0 \tag{12}$$

which is related by setting $V' = -y$ to the fourth-order non-autonomous ordinary differential equation

$$y^{(4)} - 18yy'' - 9y'^2 + 24y^3 - \alpha y^2 - \frac{\alpha^2}{9}y = \kappa x + \beta \tag{13}$$

given in the Cosgrove classification [6] and defining a new transcendent.

In the case of the stationary reduction $z = V(x - ct)$, equation (11) can be identified with equation (13) by setting $y = -V' + \frac{c}{9}, \alpha = 8c$, in the particular case $\kappa = 0$. As proved by Cosgrove, its general solution is given in terms of hyperelliptic functions of genus two and, in the degenerate case $\alpha = \beta = 0$, it can be solved in terms of a pair of elliptic functions of genus one.

On the other hand, setting $z = \partial_{x_1}\operatorname{Log}\tau_0(\vec{x})|_{x_2=x_4=\cdots=0}$ in the system (3)–(4), it becomes the $(2 + 1)$-dimensional PDE, called the CKP equation:

$$9z_{x_1,x_5} - 5z_{2x_3} + \left(-5z_{2x_1,x_3} - 15z_{x_1}z_{x_3} + z_{5x_1} + 15z_{x_1}z_{3x_1} + 15\left(z_{x_1}\right)^3 + \tfrac{45}{4}\left(z_{2x_1}\right)^2\right)_{x_1} = 0 \quad \left(z_{2x_3} \equiv z_{x_3x_3}\ldots\right). \tag{14}$$

This represents either a generalization in two space dimensions [7] of the Kaup–Kupershmidt (KK) equation by setting $x_1 = x, x_3 = y, x_5 = t$ or a generalization in two space dimensions of the bidirectional KK (bKK) equation [8] by setting $x_1 = x, x_5 = y, x_3 = t$.

The lower dimension reductions are

- $\partial_{x_3}\tau_0 = 0, x_1 \equiv x, x_5 \equiv 9t$

$$D_x^4(f\cdot f) + 6fg = 0 \tag{15}$$

$$\left(D_x^6 + 16D_xD_t\right)(f\cdot f) - 90D_x^2(f\cdot g) = 0 \tag{16}$$

equivalent to the fifth-order potential KK equation [9]

$$z_t + z_{xxxxx} + 15z_xz_{xxx} + 15(z_x)^3 + \tfrac{45}{4}(z_{xx})^2 = 0 \qquad z = \partial_x \operatorname{Log} f. \tag{17}$$

- $\partial_{x_5}\tau_0 = 0, x_1 \equiv x, x_3 \equiv t$

$$\left(D_x^4 - 4D_xD_t\right)(f\cdot f) + 6fg = 0 \tag{18}$$

$$\left(D_x^6 - 20D_x^3D_t - 80D_t^2\right)(f\cdot f) - 45D_x^2(f\cdot g) = 0 \tag{19}$$

equivalent to the sixth-order bKK equation [8]

$$\left(z_{xxxxx} + 15z_xz_{xxx} + 15(z_x)^3 - 15z_xz_t - 5z_{xxt} + \tfrac{45}{4}(z_{xx})^2\right)_x - 5z_{tt} = 0. \tag{20}$$

In this letter, we analyse the dynamics of the soliton solutions of the equations (11) and (20) which are especially interesting as they may describe the head-on collisions of solitary waves of different shape and type. We also derive the analytical expression of their N-soliton solutions in terms of a Grammian.

2. *N*-soliton solutions of the bSH equation

From the Bäcklund transformation (BT)

$$p_t - \left(p_{xx} - \frac{3}{4}\frac{p_x^2}{p} + \frac{3}{2}pp_x + 3z_xp + \frac{1}{4}p^3\right)_x = 0 \qquad p = Z - z \tag{21}$$

$$\begin{aligned}\frac{1}{2}p_{xxxx} &+ \frac{5}{3}z_{xxx}p - \frac{p_xp_{xxx}}{p} + \frac{3}{2}pp_{xxx} + 2z_{xx}p_x + 2z_{xx}p^2 - \frac{3}{4}\frac{p_{xx}^2}{p} + 2z_xp_{xx} + \frac{9}{4}\frac{p_x^2p_{xx}}{p^2}\\ &+ p_xp_{xx} + \frac{7}{4}p^2p_{xx} + 2z_x^2p - z_x\frac{p_x^2}{p} + 4z_xpp_x + z_xp^3 - \frac{15}{16}\frac{p_x^4}{p^3} + \frac{15}{8}pp_x^2\\ &+ p^3p_x + \frac{4}{3}z_tp + \frac{1}{16}p^5 - \lambda p = 0\end{aligned} \tag{22}$$

where z and Z are two different solutions of equation (11), we have shown in [10] that this equation possesses N-soliton solutions which are linked, as for the KK equation, to a Grammian

$$z^{(N)} = \partial_x \operatorname{Log} f^{(N)} \tag{23}$$

$$f^{(N)} = \det\left[\int \phi_i\phi_j\,\mathrm{d}x\right]_{1\leqslant i,j\leqslant N}. \tag{24}$$

Here ϕ_i satisfies the fourth-order Lax pair [2]

$$\left(\partial_x^4 + 4u\partial_x^2 + 4u_x\partial_x + 2u_{xx} + 4u^2 + v\right)\psi = \lambda\psi \tag{25}$$

$$\left(\partial_x^3 + 3u\partial_x + \tfrac{3}{2}u_x\right)\psi = \partial_t\psi \qquad (u, v \text{ solution of the system (9)–(10)}) \tag{26}$$

for $u = v = 0$, $\lambda \equiv \lambda_i$. Moreover, following a remark by Satsuma and Hirota [2], we verify that equation (11) may also describe the collision of solitary waves of different type. Indeed, as the dispersion relation associated with the linear part of equation (11) is of second degree in ω

$$F(\omega, k) \equiv 8\omega^2 - k^6 + 2\omega k^3 = 0 \tag{27}$$

we have two types of solitary waves depending on the direction of propagation, with speed $v_+ = k^2/2$ and $v_- = -k^2/4$.

The solitary wave propagating in the negative direction possesses a KdV-like profile given by

$$u_- = \partial_x^2 \operatorname{Log}(1 + \mathrm{e}^{\theta_-}) \qquad \theta_- = kx + \tfrac{1}{4}k^3 t + \delta_- \tag{28}$$

and corresponds to the case where the system (9)–(10) degenerates ($v \equiv 0$) into the KdV equation, while the solitary wave propagating in the positive direction is

$$u_+ = \partial_x^2 \operatorname{Log}\left(1 + 2\,\mathrm{e}^{\theta_+} + \tfrac{1}{2}\,\mathrm{e}^{2\theta_+}\right) \qquad \theta_+ = kx - \tfrac{1}{2}k^3 t + \delta_+. \tag{29}$$

Therefore, at the level of the two-soliton solution

$$u_{12} = \partial_x^2 \operatorname{Log} f_{12} \tag{30}$$

we have not only the solution described by expression (24), for $N = 2$,

$$\begin{aligned} f_{12} \equiv f_{12}^{(++)} &= 1 + 2(\mathrm{e}^{\theta_{1,+}} + \mathrm{e}^{\theta_{2,+}}) + \frac{1}{2}(\mathrm{e}^{2\theta_{1,+}} + \mathrm{e}^{2\theta_{2,+}}) + 4B_{12}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,+}} \\ &\quad + A_{12}(\mathrm{e}^{2\theta_{1,+}+\theta_{2,+}} + \mathrm{e}^{\theta_{1,+}+2\theta_{2,+}}) + \frac{A_{12}^2}{4}\,\mathrm{e}^{2(\theta_{1,+}+\theta_{2,+})} \end{aligned} \tag{31}$$

$$A_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \qquad B_{12} = \frac{\left(k_1^4 + k_2^4\right)}{(k_1 + k_2)^2\left(k_1^2 + k_2^2\right)} \tag{32}$$

but also the usual expression for the two-soliton solution of the KdV equation

$$f_{12} \equiv f_{12}^{(--)} = 1 + \mathrm{e}^{\theta_{1,-}} + \mathrm{e}^{\theta_{2,-}} + A_{12}\,\mathrm{e}^{\theta_{1,-}+\theta_{2,-}} \tag{33}$$

and a new expression

$$f_{12} \equiv f_{12}^{(+-)} = 1 + 2\,\mathrm{e}^{\theta_{1,+}} + \mathrm{e}^{\theta_{2,-}} + \frac{1}{2}\,\mathrm{e}^{2\theta_{1,+}} + 2\tilde{A}_{12}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,-}} + \frac{\tilde{A}_{12}^2}{2}\,\mathrm{e}^{2\theta_{1,+}+\theta_{2,-}} \tag{34}$$

$$\tilde{A}_{12} = \frac{2k_1^2 - 2k_1k_2 + k_2^2}{2k_1^2 + 2k_1k_2 + k_2^2} \tag{35}$$

representing the head-on collision of KdV and cKdV solitary waves (figure 1).

At the level of the three-soliton solution

$$u_{123} = \partial_x^2 \operatorname{Log} f_{123}. \tag{36}$$

We have four different types: the solution given by expression (24), for $N = 3$, and representing the collision of three cKdV-type waves like equation (29); the three-soliton solution of the KdV equation; and two new solutions representing the head-on collision of a KdV wave with a cKdV-like wave, in the presence of either a KdV-like wave (figure 2) or a cKdV-like wave (figure 3).

The explicit expressions of these solutions are given in appendix A.

3. *N*-soliton solution of the bKK equation

The (2+1)-dimensional equation (14) arises from the compatibility condition of the following linear operators

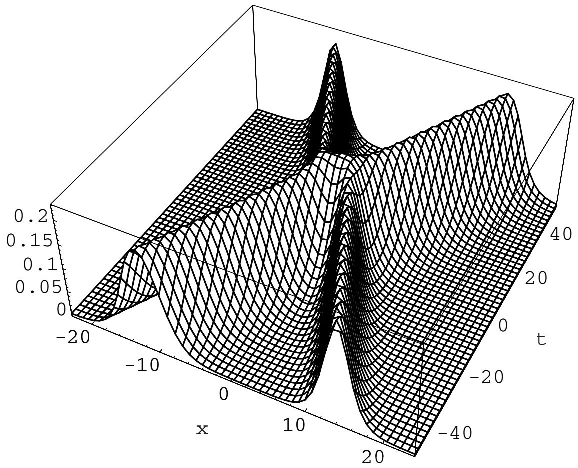

Figure 1. $u_{12}^{(+-)}$ ($k_1 = 0.7, k_2 = 0.9$).

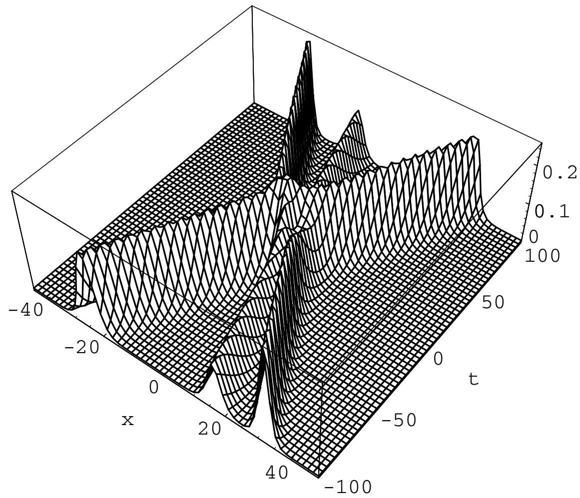

Figure 2. $u_{123}^{(+--)}$ ($k_1 = 0.7, k_2 = 0.7, k_3 = 1$).

$$L \equiv \partial_{x_3} - \partial_{x_1}^3 - 3u\partial_{x_1} - \tfrac{3}{2}u_{x_1} \tag{37}$$

$$M \equiv \partial_{x_5} - \partial_{x_1}^5 - 5u\partial_{x_1}^3 - \tfrac{15}{2}u_{x_1}\partial_{x_1}^2 - \left(5u^2 + \tfrac{35}{6}u_{2x_1} + \tfrac{15}{9}z_{x_3}\right)\partial_{x_1} \\ - 5uu_{x_1} - \tfrac{5}{3}u_{3x_1} - \tfrac{15}{18}u_{x_3} \qquad u = z_{x_1} \tag{38}$$

$$[L, M] = 0 \iff E_{\mathrm{CKP}}(z) = 0 \tag{39}$$

which act on the wavefunction $\phi(x_1, x_3, x_5)$. Considering the following reduction (u independent of x_3)

$$u(x_1, x_3, x_5) \equiv u(x_1, x_5) \qquad \phi(x_1, x_3, x_5) = \mathrm{e}^{\lambda x_3}\phi(x_1, x_5) \\ x_5 \equiv 9t \qquad x_1 \equiv x \qquad \lambda \text{ constant} \tag{40}$$

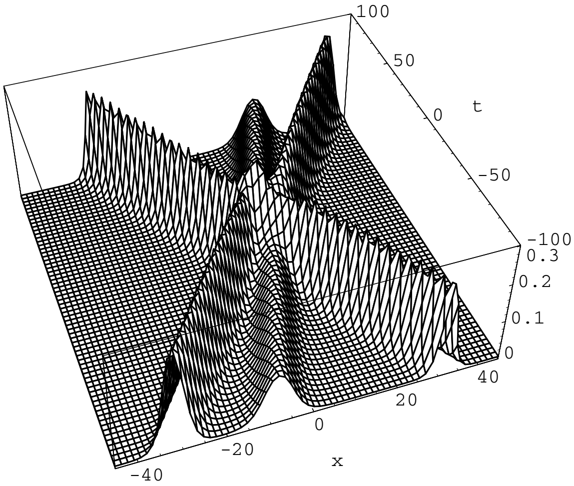

Figure 3. $u_{123}^{(++-)}$ $(k_1 = 0.55, k_2 = 0.8, k_3 = 1)$.

we recover the third-order Lax pair [9] of the KK equation

$$\left(\partial_x^3 + 3u\partial_x + \tfrac{3}{2}u_x\right)\phi = \lambda\phi \tag{41}$$

$$\left(\partial_x^5 + 5u\partial_x^3 + \tfrac{15}{2}u_x\partial_x^2 + \left(5u^2 + \tfrac{35}{6}u_{xx}\right)\partial_x + 5uu_x + \tfrac{5}{3}u_{xxx}\right)\phi = \tfrac{1}{9}\partial_t\phi \tag{42}$$

$$[L, M] = 0 \iff u_t + \left(u_{xxxx} + 15uu_{xx} + 15u^3 + \tfrac{45}{4}(u_x)^2\right)_x = 0. \tag{43}$$

When u is independent of x_5

$$u(x_1, x_3, x_5) \equiv u(x_1, x_3) \qquad \phi(x_1, x_3, x_5) = \mathrm{e}^{\lambda x_5}\phi(x_1, x_3) \qquad x_3 \equiv t \qquad x_1 \equiv x \tag{44}$$

we obtain the Lax pair [8] of the potential bKK equation (20)

$$\begin{aligned}&\left(\partial_x^5 + 5u\partial_x^3 + \tfrac{15}{2}u_x\partial_x^2 + \left(5u^2 + \tfrac{35}{6}u_{xx} + \tfrac{15}{9}z_t\right)\partial_x\right.\\ &\qquad\left. + 5uu_x + \tfrac{5}{3}u_{xxx} + \tfrac{15}{18}u_t\right)\phi = \lambda\phi \qquad u = z_x\end{aligned} \tag{45}$$

$$\left(\partial_x^3 + 3u\partial_x + \tfrac{3}{2}u_x\right)\phi = \partial_t\phi \tag{46}$$

$$[L, M] = 0 \iff E_{\mathrm{pbKK}}(z) = 0. \tag{47}$$

Considering the Darboux transformation (DT) [11]

$$Z = \partial_x \operatorname{Log} f + z \qquad f = \int^x \phi^2 \tag{48}$$

we obtain, by elimination of ϕ between the DT (48) and the linear equations (45)–(46) the following BT

$$p_t - \left(p_{xx} - \frac{3p_x^2}{4p} + \frac{3}{2}pp_x + 3z_x p + \frac{1}{4}p^3\right)_x = 0 \qquad p = Z - z \tag{49}$$

$$\frac{1}{2}p_{xxxx} + \frac{5}{3}z_{xxx}p - \frac{5}{4}\frac{p_x p_{xxx}}{p} + \frac{3}{2}pp_{xxx} + \frac{5}{4}z_{xx}p_x + \frac{5}{4}z_{xx}p^2 - \frac{5}{8}\frac{p_{xx}^2}{p} + \frac{5}{2}z_x p_{xx} + \frac{5}{2}\frac{p_x^2 p_{xx}}{p^2}$$

$$+\frac{5}{4}p_x p_{xx} + \frac{5}{4}p^2 p_{xx} + \frac{5}{2}z_x^2 p - \frac{15}{8}z_x\frac{p_x^2}{p} + \frac{15}{4}z_x p p_x + \frac{5}{8}z_x p^3 - 35\frac{p_x^4}{p^3}$$
$$+\frac{25}{16}pp_x^2 + \frac{5}{8}p^3 p_x - \frac{5}{8}\frac{p_x^3}{p} + \frac{5}{6}z_t p + \frac{1}{32}p^5 - \lambda = 0. \qquad (50)$$

Here, equation (49) is identical to equation (21). Therefore, as for the PDE (11), we easily deduce the nonlinear superposition formula (NLSF)

$$f_{12} = f_0 \begin{vmatrix} \int^x \phi_1^2 & \int^x \phi_1\phi_2 \\ \int^x \phi_1\phi_2 & \int^x \phi_2^2 \end{vmatrix} \qquad (51)$$

where ϕ_1 and ϕ_2 are solutions of the Lax pair (45)–(46) with respective spectral parameters λ_1 and λ_2 and the potential $z_0 = \partial_x \operatorname{Log} f_0$ solution of the potential bKK equation (20). If we choose as the seed solution $f_0 = 1$, $z_{12} = \partial_x \operatorname{Log} f_{12}$ represents the two-soliton solution of equation (20) depending on the two different parameters λ_1 and λ_2. The NLSF for the N-soliton solution is therefore as for the KK equation [12]

$$f^{(N)} \equiv \det\left[\int \phi_i\phi_j \,\mathrm{d}x\right]_{1\leqslant i,j\leqslant N} \qquad (52)$$

where ϕ_i is the vacuum wavefunction, the solution of the Lax pair (45)–(46) for $z = 0$, $\lambda = \lambda_i$.

We can therefore easily construct the soliton solutions starting from the vacuum wavefunction given by

$$\phi(x,t) = \sum_{i=1}^{5} A_i\,\mathrm{e}^{p_i x + p_i^3 t} \qquad p_i^5 = \lambda \qquad p_i - p_j \neq 0 \quad \text{for} \quad i \neq j \quad (53)$$

where A_i are constant.

Choosing the particular solution

$$\phi(x,t) = A\,\mathrm{e}^{px+p^3t} + B\,\mathrm{e}^{qx+q^3t} \qquad p \neq q \qquad (54)$$

we have on p and q, taking into account that $p^5 = q^5$, the following conditions

$$p^4 + p^3q + p^2q^2 + pq^3 + q^4 = 0 \qquad (55)$$

and

$$f = \int^x \phi^2 \simeq 1 + 4\frac{B}{A}\frac{p}{p+q}\,\mathrm{e}^{(q-p)x+(q^3-p^3)t} + \frac{B^2}{A^2}\frac{p}{q}\,\mathrm{e}^{2(q-p)x+2(q^3-p^3)t}. \qquad (56)$$

(The sign $\simeq$ means that the previous relation is valid up to an exponential factor linear in x and t.)

Setting in equation (55) $p = ak$ and $q = (1+a)k$, a solves the fourth degree algebraic equation

$$5a^4 + 10a^3 + 10a^2 + 5a + 1 = 0. \qquad (57)$$

Therefore, the ratio $(q^3 - p^3)/(p - q) = -(q^2 + qp + p^2)$ may have two distinct values and the speed of the one-soliton is either

$$v_+ = \frac{\sqrt{5}(3+\sqrt{5})}{10}k^2 > 0 \qquad \text{or} \qquad v_- = \frac{\sqrt{5}(-3+\sqrt{5})}{10}k^2 < 0 \qquad (58)$$

such that we have two different profiles (figure 4)

$$u_\pm^{(1)} = \partial_x^2 \operatorname{Log} f \equiv \partial_x^2 \operatorname{Log}\left(1 + 4\,\mathrm{e}^{\theta_\pm} + \frac{3\pm\sqrt{5}}{2}\,\mathrm{e}^{2\theta_\pm}\right)$$
$$\theta_\pm = k(x - v_\pm t) + \delta \qquad \delta = \operatorname{Log}\left(\frac{B}{A}\frac{p}{p+q}\right). \qquad (59)$$

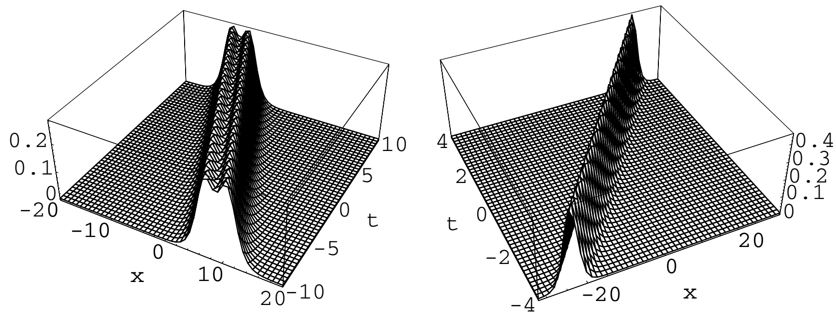

Figure 4. $u_{-}^{(1)}$ and $u_{+}^{(1)}$ for $k = 1$.

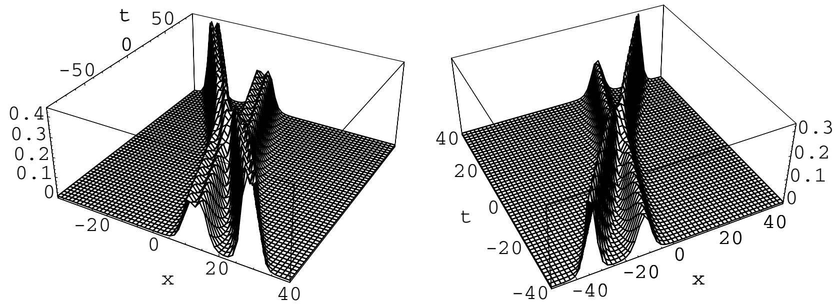

Figure 5. $u_{12}^{(--)}$ $(k_1 = 0.9, k_2 = 1.25)$ and $u_{12}^{(++)}$ $(k_1 = 0.6, k_2 = 0.85)$.

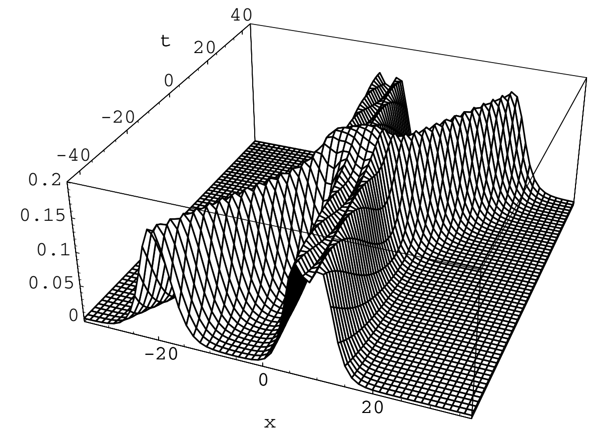

Figure 6. $u_{12}^{(+-)}$ $(k_1 = 0.6, k_2 = 0.75)$.

These two values for the speed agree with those we can obtain by solving the dispersion relation associated with the linear part of equation (20)

$$F(\omega, k) \equiv 5\omega^2 - k^6 + 5\omega k^3 = 0 \tag{60}$$

which is of second degree in ω.

For the two-soliton solution we have three different possibilities, two of these representing overtaking collisions (figure 5) and the third the head-on collision of solitary waves (figure 6).

4. Conclusion

The two integrable PDEs considered here are interesting not only because of the dynamics of their soliton solutions, but also because of their relationship with a generalized Hénon–Heiles Hamiltonian. Written in the form of two coupled KdV systems, the equations (11) and (20) are related by stationary reduction to three Liouville integrable cases of the extended quartic Hénon–Heiles Hamiltonian. The new links established in [13] between those two coupled KdV systems and the others given by Baker [14] could help to integrate explicitly with functions possessing the Painlevé property some cases, which until now, have only been integrated in the Liouville sense.

Acknowledgment

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Appendix A. Explicit expressions of the three-soliton solutions for equation (11)

$$\begin{aligned} f_{123}^{(---)} &= 1 + e^{\theta_{1,-}} + e^{\theta_{2,-}} + e^{\theta_{3,-}} + A_{12}\, e^{\theta_{1,-}+\theta_{2,-}} + A_{13}\, e^{\theta_{1,-}+\theta_{3,-}} \\ &\quad + A_{23}\, e^{\theta_{2,-}+\theta_{3,-}} + A_{12}A_{13}A_{23}\, e^{\theta_{1,-}+\theta_{2,-}+\theta_{3,-}} \end{aligned} \tag{61}$$

$$\begin{aligned} f_{123}^{(+--)} &= 1 + 2\, e^{\theta_{1,+}} + e^{\theta_{2,-}} + e^{\theta_{3,-}} + \frac{1}{2} e^{2\theta_{1,+}} + 2\tilde{A}_{12}\, e^{\theta_{1,+}+\theta_{2,-}} + 2\tilde{A}_{13}\, e^{\theta_{1,+}+\theta_{3,-}} + A_{23}\, e^{\theta_{2,-}+\theta_{3,-}} \\ &\quad + \frac{\tilde{A}_{12}^2}{2} e^{2\theta_{1,+}+\theta_{2,-}} + \frac{\tilde{A}_{13}^2}{2} e^{2\theta_{1,+}+\theta_{3,-}} + 2A_{23}\tilde{A}_{12}\tilde{A}_{13}\, e^{\theta_{1,+}+\theta_{2,-}+\theta_{3,-}} \\ &\quad + \frac{1}{2} A_{23}\tilde{A}_{12}^2\tilde{A}_{13}^2\, e^{2\theta_{1,+}+\theta_{2,-}+\theta_{3,-}} \end{aligned} \tag{62}$$

$$\begin{aligned} f_{123}^{(++-)} &= 1 + 2\, e^{\theta_{1,+}} + 2\, e^{\theta_{2,+}} + e^{\theta_{3,-}} + \frac{1}{2} e^{2\theta_{1,+}} + \frac{1}{2} e^{2\theta_{2,+}} + 4B_{12}\, e^{\theta_{1,+}+\theta_{2,+}} + 2\tilde{A}_{13}\, e^{\theta_{1,+}+\theta_{3,-}} \\ &\quad + 2\tilde{A}_{23}\, e^{\theta_{2,+}+\theta_{3,-}} + A_{12}(e^{2\theta_{1,+}+\theta_{2,+}} + e^{\theta_{1,+}+2\theta_{2,+}}) + \frac{\tilde{A}_{13}^2}{2} e^{2\theta_{1,+}+\theta_{3,-}} + \frac{\tilde{A}_{23}^2}{2} e^{\theta_{2,+}+2\theta_{3,-}} \\ &\quad + 4B_{12}\tilde{A}_{13}\tilde{A}_{23}\, e^{\theta_{1,+}+\theta_{2,+}+\theta_{3,-}} + \frac{A_{12}^2}{4} e^{2\theta_{1,+}+2\theta_{2,+}} + A_{12}\tilde{A}_{13}\tilde{A}_{23}\, e^{\theta_{1,+}+\theta_{2,+}+\theta_{3,-}} \\ &\quad \times (\tilde{A}_{13}\, e^{\theta_{1,+}} + \tilde{A}_{23}\, e^{\theta_{2,+}}) + \frac{1}{4} A_{12}^2\tilde{A}_{13}^2\tilde{A}_{23}^2\, e^{2\theta_{1,+}+2\theta_{2,+}+\theta_{3,-}} \end{aligned} \tag{63}$$

$$\begin{aligned} f_{123}^{(+++)} &= 1 + 2(e^{\theta_{1,+}} + e^{\theta_{2,+}} + e^{\theta_{3,+}}) + \frac{1}{2}(e^{2\theta_{1,+}} + e^{2\theta_{2,+}} + e^{2\theta_{3,+}}) \\ &\quad + 4(B_{12}\, e^{\theta_{1,+}+\theta_{2,+}} + B_{13}\, e^{\theta_{1,+}+\theta_{3,+}} + B_{23}\, e^{\theta_{2,+}+\theta_{3,+}}) + A_{12}(e^{2\theta_{1,+}+\theta_{2,+}} + e^{\theta_{1,+}+2\theta_{2,+}}) \\ &\quad + A_{13}(e^{2\theta_{1,+}+\theta_{3,+}} + e^{\theta_{1,+}+2\theta_{3,+}}) + A_{23}(e^{2\theta_{2,+}+\theta_{3,+}} + e^{\theta_{2,+}+2\theta_{3,+}}) \\ &\quad + C_{123}\, e^{\theta_{1,+}+\theta_{2,+}+\theta_{3,+}} + \frac{A_{12}^2}{4} e^{2\theta_{1,+}+2\theta_{2,+}} + \frac{A_{13}^2}{4} e^{2\theta_{1,+}+2\theta_{3,+}} + \frac{A_{23}^2}{4} e^{2\theta_{2,+}+2\theta_{3,+}} \\ &\quad + 2\, e^{\theta_{1,+}+\theta_{2,+}+\theta_{3,+}}(A_{12}A_{13}B_{23}\, e^{\theta_{1,+}} + A_{12}B_{13}A_{23}\, e^{\theta_{2,+}} + B_{12}A_{13}A_{23}\, e^{\theta_{3,+}}) \\ &\quad + \frac{1}{2} A_{12}A_{13}A_{23}\, e^{\theta_{1,+}+\theta_{2,+}+\theta_{3,+}}(A_{12}\, e^{\theta_{1,+}+\theta_{2,+}} + A_{13}\, e^{\theta_{1,+}+\theta_{3,+}} + A_{23}\, e^{\theta_{2,+}+\theta_{3,+}}) \\ &\quad + \frac{1}{8} A_{12}^2A_{13}^2A_{23}^2\, e^{2(\theta_{1,+}+\theta_{2,+}+\theta_{3,+})} \end{aligned} \tag{64}$$

$$A_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \qquad \tilde{A}_{ij} = \frac{2k_i^2 - 2k_ik_j + k_j^2}{2k_i^2 + 2k_ik_j + k_j^2} \qquad B_{ij} = \frac{k_i^4 + k_j^4}{(k_i + k_j)^2\left(k_i^2 + k_j^2\right)}$$
$$C_{123} = 8\frac{k_1^4k_2^4\left(k_1^4 + k_2^4\right) + k_1^4k_3^4\left(k_1^4 + k_3^4\right) + k_2^4k_3^4\left(k_2^4 + k_3^4\right) - 6k_1^4k_2^4k_3^4}{(k_1 + k_2)^2\left(k_1^2 + k_2^2\right)(k_1 + k_3)^2\left(k_1^2 + k_3^2\right)(k_2 + k_3)^2\left(k_2^2 + k_3^2\right)} \tag{65}$$

Appendix B. Explicit expression of the two-soliton solution for equation (20)

$$u_{12}^{(\pm\pm)} = \partial_x^2 \operatorname{Log} f_{12}^{(\pm\pm)} \tag{66}$$

$$f_{12}^{(\pm\pm)} = 1 + 4(\mathrm{e}^{\theta_{1,\pm}} + \mathrm{e}^{\theta_{2,\pm}}) + \frac{3 \pm \sqrt{5}}{2}(\mathrm{e}^{2\theta_{1,\pm}} + \mathrm{e}^{2\theta_{2,\pm}}) + B_{12}^{(\pm\pm)}\,\mathrm{e}^{\theta_{1,\pm}+\theta_{2,\pm}}$$
$$+ 4A_{12}^{(\pm\pm)}(\mathrm{e}^{2\theta_{1,\pm}+\theta_{2,\pm}} + \mathrm{e}^{\theta_{1,\pm}+2\theta_{2,\pm}}) + A_{12}^{(\pm\pm)2}\,\mathrm{e}^{2(\theta_{1,\pm}+\theta_{2,\pm})} \tag{67}$$

$$A_{12}^{(\pm\pm)} = \frac{3 \pm \sqrt{5}}{2}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \frac{k_1^2 - \frac{1\mp\sqrt{5}}{2}k_1k_2 + k_2^2}{k_1^2 + \frac{1\mp\sqrt{5}}{2}k_1k_2 + k_2^2} \tag{68}$$

$$B_{12}^{(\pm\pm)} = \frac{16\left(k_1^4 - \frac{3\mp\sqrt{5}}{4}k_1^2k_2^2 + k_2^4\right)}{(k_1 + k_2)^2\left(k_1^2 + \frac{1\mp\sqrt{5}}{2}k_1k_2 + k_2^2\right)} \tag{69}$$

$$u_{12}^{(+-)} = \partial_x^2 \operatorname{Log} f_{12}^{(+-)} \tag{70}$$

$$f_{12}^{(+-)} = 1 + 4(\mathrm{e}^{\theta_{1,+}} + \mathrm{e}^{\theta_{2,-}}) + \frac{3 + \sqrt{5}}{2}\,\mathrm{e}^{2\theta_{1,+}} + \frac{3 - \sqrt{5}}{2}\,\mathrm{e}^{2\theta_{2,-}} + B_{12}^{(+-)}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,-}}$$
$$+ 4A_{12}^{(+-)}\left(\frac{3 + \sqrt{5}}{2}\,\mathrm{e}^{2\theta_{1,+}+\theta_{2,-}} + \frac{3 - \sqrt{5}}{2}\,\mathrm{e}^{\theta_{1,+}+2\theta_{2,-}}\right) + A_{12}^{(+-)2}\,\mathrm{e}^{2(\theta_{1,+}+\theta_{2,-})} \tag{71}$$

$$A_{12}^{(+-)} = \frac{k_1^4 - \frac{5-\sqrt{5}}{2}k_1^3k_2 + 3\frac{3-\sqrt{5}}{2}k_1^2k_2^2 - (5 - 2\sqrt{5})k_1k_2^3 + \frac{7-3\sqrt{5}}{2}k_2^4}{k_1^4 + \frac{5-\sqrt{5}}{2}k_1^3k_2 + 3\frac{3-\sqrt{5}}{2}k_1^2k_2^2 + (5 - 2\sqrt{5})k_1k_2^3 + \frac{7-3\sqrt{5}}{2}k_2^4} \tag{72}$$

$$B_{12}^{(+-)} = \frac{16\left(k_1^4 + \frac{3-\sqrt{5}}{4}k_1^2k_2^2 + \frac{7-3\sqrt{5}}{2}k_2^4\right)}{k_1^4 + \frac{5-\sqrt{5}}{2}k_1^3k_2 + 3\frac{3-\sqrt{5}}{2}k_1^2k_2^2 + (5 - 2\sqrt{5})k_1k_2^3 + \frac{7-3\sqrt{5}}{2}k_2^4} \tag{73}$$

$$\theta_{1,\pm} = k_1(x - v_{1,\pm}t) + \delta_{1,\pm} \qquad v_{1,\pm} = \frac{\sqrt{5}(\pm 3 + \sqrt{5})}{10}k_1^2 \tag{74}$$

$$\theta_{2,\pm} = k_2(x - v_{2,\pm}t) + \delta_{2,\pm} \qquad v_{2,\pm} = \frac{\sqrt{5}(\pm 3 + \sqrt{5})}{10}k_2^2 \tag{75}$$

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